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The Value Function of a Differential Game with Simple Motions and an Integro – Terminal Payoff Functional ⋆

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Abstract: An antagonistic positional differential game of two persons is considered. The dynamics of the system is described by a differential equation with simple motions, and the payoff functional is integro - terminal. For the case when the terminal function and the Hamiltonian are piecewise linear, and the dimension of the state space is two, a finite algorithm for the exact construction of the value function is proposed.

Keywords: algorithms, differential equations; differential games; feedback control; optimal control; piecewise linear analysis; value function

1. INTRODUCTION

Differential games with simple motions are one of the simple models of conflict-controlled systems. Dynamics in such games depends only on the players' controls. The solutions of such games, which are of independent interest, are also used in numerical algorithms for solving differential games of a general type. Despite the simplicity of the dynamics, solutions to such games are known only in special cases, in the general case finding solutions is not an easy task.

Differential games with simple motions were studied by many authors, in particular, Petrosjan (1993); Pachter and Yavin (1986); Ukhobotov (1991); Kamneva and Patsko (2016). The present paper deals with an antagonistic differential game with fixed termination moment. The game is considered in the context of the positional formalization developed by Krasovskii and Subbotin (1974, 1988).

In the considered game the payoff functional is given. The first player minimizes the payoff, and the second player maximizes it from a given initial point. The both players use feedback strategies. The value of the game is an optimal guaranteed result for the both players. The value function assigns an optimal guaranteed result to each initial position of the game.

The optimal feedback strategies of the players can be constructed on the basis of the value function, therefore methods for finding value functions are important in differential games.

It is known (see, Subbotin (1991, 1995)) that the value function is a minimax solution of Hamilton - Jacobi - Bellman – Isaacs equation corresponding to the considered differential game. The concept of minimax solution is equivalent to the concept of viscosity solution introduced by Crandall and Lions (1983).

In this paper we present a finite algorithm for constructing the exact minimax solution for Hamilton - Jacobi equation corresponding to differential game with simple motions and an integro – terminal payoff functional for the case of two-dimensional state space and piecewise linear input data. Presented results generalize results obtained in [Subbotin and Shagalova (1992); Shagalova (1999)].

2. THE PROBLEM STATEMENT

We consider the following antagonistic positional differential game on a bounded time interval. The motion of a controlled system is described by equation

$$\begin{array}{l} \dot{x} = u(t) + v(t),\ t \in [0, \vartheta], \quad x \in R^n, \\ u(t) \in P \subset R^n, \quad v(t) \in Q \subset R^n \end{array} \qquad (1)$$

Here t is the time, ϑ is the fixed terminal moment of the game, x is the state vector, $u(\cdot)$ and $v(\cdot)$ are controls of the first and the second players respectively. The sets P and Q are compact.

Let $(t_0, x_0) \in [0, \vartheta] \times R^n$ be an initial position. The integro-terminal payoff functional is given

$$I = I(t_0, x_0, u(\cdot), v(\cdot)) = \sigma(x(\vartheta)) + \int_{t_0}^{\vartheta} g(u(\tau), v(\tau))d\tau, \quad (2)$$

where the function $\sigma : R^n \to R$ is assumed to be Lipschitz continuous, and the function $g : P \times Q$ is continuous. The first player is trying to minimize the payoff by choosing his control, while the second player is trying to maximize the payoff.

We suppose that for the differential game under consideration the following condition is satisfied

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$$\begin{aligned} &\min_{u\in P}\max_{v\in Q}[< s, u+v > +g(u,v)] = \\ &= \max_{v\in Q}\min_{u\in P}[< s, u+v > +g(u,v)] = H(s), \quad s \in R^n \end{aligned} \tag{3}$$

where $< s, f >$ denotes the inner product of vectors s and f. The function $H(\cdot)$ defined by equality (3) will be called *the Hamiltonian* of differential game (1), (2).

It is proved in Krasovskii and Subbotin (1988) that if the condition (3) holds, then for any initial position $(t_0, x_0) \in [0, \vartheta] \times R^n$, there exists the value $\omega(t_0, x_0)$ of the game. Thus, there is the value function $\omega : [0, \vartheta] \times R^n \to R$. However, finding the value function is not an easy task, for the solution of which there is no universal method. In the case when the integrand $g(\cdot)$ is identically equal to zero, that is, the payoff function is terminal, the problem is substantially simplified. In particular, if one of the functions $H(\cdot)$ or $\sigma(\cdot)$ is convex or concave, we can write explicit formulas for the value function using known Hopf-Lax formulas (see Hopf (1965); Pshenichyi and Sagaidak (1970); Bardi and Evans (1984)). Also, in the case of not necessarily convex or concave piecewise linear $H(\cdot)$ and $\sigma(\cdot)$, when the dimension n of state space is equal to two, and the terminal payoff is positively homogeneous, that is, the function $\sigma(\cdot)$ satisfies the condition

$$\sigma(\lambda x) = \lambda\sigma(x), \quad x \in R^n, \quad \lambda \in R, \lambda > 0, \tag{4}$$

to construct the value function exactly, one can use the finite algorithm [Subbotin and Shagalova (1992); Shagalova (1999)]. The aim of this paper is to generalize this algorithm to the case of a nonzero piecewise linear integrand $g(\cdot)$.

3. THE VALUE FUNCTION AS A MINIMAX SOLUTION OF THE HAMILTON-JACOBI EQUATION

This section contains information from [Subbotin (1991, 1995)] and facts that are easy to obtain from this information, used below for the development of the algorithm for constructing the value function of differential game (1)-(3).

The value function $\omega : [0, \vartheta] \times R^n \to R$ coincides with the minimax solution of the following Cauchy problem.

$$\frac{\partial\omega(t,x)}{\partial t} + H\left(\frac{\partial\omega(t,x)}{\partial x}\right) = 0, \quad t \le \vartheta, x \in \times R^n \tag{5}$$

$$\omega(\vartheta, x) = \sigma(x), \quad x \in R^n. \tag{6}$$

The minimax solution of problem (5), (6) exists and is unique.

We further assume that the terminal function $\sigma(\cdot)$ is positively homogeneous, that is, it satisfies condition (4).

The Hamiltonian $H(\cdot)$ is defined by equality (3) and, as is not difficult to see, is not positively homogeneous.

Let us define functions

$$H^*(s,r) = \begin{cases} |r|H(\dfrac{s}{|r|}) & r \ne 0, \\ \lim\limits_{r\downarrow 0} rH(\dfrac{s}{r}) & r = 0, \end{cases} \quad (s,r) \in R^n \times R, \tag{7}$$

$$\sigma^\sharp(x,y) = \sigma(x) + y, \quad x \in R^n, y \in R. \tag{8}$$

We assume that the limit in (7) exists.

Consider the Cauchy problem for the Hamilton-Jacobi equation with the Hamiltonian $H^*(\cdot)$ positively homogeneous with respect to the variable $\overline{s} = (s, r)$.

$$\begin{aligned} &\frac{\partial u(t,x,y)}{\partial t} + H^*\left(\frac{\partial u(t,x,y)}{\partial x}, \frac{\partial u(t,x,y)}{\partial y}\right) = 0, \\ &t \le \vartheta, \ (x,y) \in R^n \times R \end{aligned} \tag{9}$$

$$u(\vartheta, x, y) = \sigma^\sharp(x,y), \quad x \in R^n, y \in R. \tag{10}$$

The following assertion is valid.

Theorem 1. The function $\omega(t,x)$ is a minimax solution of problem (5), (6) if and only if the function $u(t,x,y) = \omega(t,x) + y$ is the minimax solution of problem (9), (10).

Thus, the problem of finding the value function for a differential game with an integro-terminal payoff functional reduces to solving the Hamilton-Jacobi equation with a positively homogeneous Hamiltonian. The dimension of the state space is increased by one in this case.

If the Hamiltonian $H^*(\cdot)$ satisfies the Lipschitz condition, then the minimax solution $u(t,x,y)$ satisfies the relation

$$u(t,x,y) = (\vartheta - t)u\left(0, \frac{x}{\vartheta - t}, \frac{y}{\vartheta - t}\right) \ x \in R^n, y \in R. \tag{11}$$

Using relation (11), we can replace problem (9), (10) with the reduced problem of finding the function

$$\varphi(x,y) = u(0,x,y) \quad x \in R^n, y \in R. \tag{12}$$

The function $\varphi(\cdot)$ is a minimax solution of the first order PDE

$$\begin{aligned} &H^*\left(\frac{\partial\varphi(x,y)}{\partial x}, \frac{\partial\varphi(x,y)}{\partial y}\right) + < \frac{\partial\varphi(x,y)}{\partial x}, x > + \\ &+ \frac{\partial\varphi(x,y)}{\partial y} \cdot y - \varphi(x,y) = 0, \quad x \in R^n, y \in R, \end{aligned} \tag{13}$$

which is considered along with the limit relation

$$\lim_{\alpha\downarrow 0} \alpha\varphi\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) = \sigma^\sharp(x,y), \quad x \in R^n, y \in R. \tag{14}$$

The minimax solution of (13) is the continuous function satisfying the pair of differential inequalities. These inequalities can be written in different equivalent forms. It is convenient for us to write these inequalities in the following form.

$$\begin{aligned} &H^*(l,m) + < l, x > + m \cdot y \le \varphi(x,y), \\ &x \in R^n, y \in R, \quad (l,m) \in D^-\varphi(x,y), \end{aligned} \tag{15}$$

$$\begin{aligned} &H^*(l,m) + < l, x > + m \cdot y \ge \varphi(x,y), \\ &x \in R^n, y \in R, \quad (l,m) \in D^+\varphi(x,y), \end{aligned} \tag{16}$$

where the sets $D^-\varphi(x,y)$ and $D^+\varphi(x,y)$ are respectively the subdifferential and the superdifferential of function $\varphi(\cdot)$ at the point (x,y).

4. THE ALGORITHM FOR THE EXACT CONSTRUCTION OF THE VALUE FUNCTION

In the case when the dimension of the state space is two, and the terminal function $\sigma(\cdot)$ and the integrand $g(\cdot)$ are piecewise linear, the value function $\omega(\cdot)$ of differential game(1)-(3) is piecewise linear and can be constructed exactly. Here we describe the algorithm for constructing the function $\varphi(\cdot)$, knowing which, it is possible to receive function $\omega(\cdot)$ by means of relation (11) and Theorem 1.

4.1 Representation of the Limit Function

Let

$$y_+ = \max\{0; y\}, \quad y_- = \min\{0; y\}$$
$$\sigma_+(x) = \max\{0; \sigma(x)\}, \quad \sigma_-(x) = \min\{0; \sigma(x)\},$$
$$\sigma^\sharp_+(x,y) = \sigma_+(x) + y_+, \quad \sigma^\sharp_-(x,y) = \sigma_-(x) + y_-,$$

where $y \in R, \quad x \in R^n$.

It is not difficult to see that limit function $\sigma^\sharp(\cdot)$ (8) can be presented in the form

$$\sigma^\sharp(x,y) = \sigma^\sharp_+(x,y) + \sigma^\sharp_-(x,y), \quad x \in R^n,\, y \in R. \tag{17}$$

Moreover, the following representation holds for the solution $\varphi(\cdot)$ of problem (13), (14)

$$\varphi(x,y) = \varphi_+(x,y) + \varphi_-(x,y), \quad x \in R^n,\, y \in R, \tag{18}$$

where $\varphi_+(\cdot)$ and $\varphi_-(\cdot)$ are solutions of the problem (13), (14) corresponding to the limit functions $\sigma^\sharp_+(\cdot)$ and $\sigma^\sharp_-(\cdot)$ respectively. Under assumptions described below, the construction of functions $\varphi_+(\cdot)$ and $\varphi_-(\cdot)$ does not differ in essence.

4.2 Assumptions

The algorithm is developed under the following assumptions.

A1. The integrand $g(\cdot)$ has the form

$$g(u,v) = g_1(u) + g_2(v), \quad u \in R^2,\, v \in R^2, \tag{19}$$

where $g_1 : R^2 \to R$ and $g_1 : R^2 \to R$ are continuous piecewise linear functions that are formed by "sewing together" a finite number of linear functions. So, their sum $g(\cdot)$ is a continuous piecewise-linear function too.

A2. Sets P and Q are polyhedrons.

It follows from (3) that the Hamiltonian $H(\cdot)$ of differential game (1)-(3) is also piecewise linear and is formed by "sewing together" a finite number of linear functions.

$$\begin{aligned} &H^i(s) =< h^i, s > + p^i, \quad i \in \overline{1, n_H}, \\ &h^i \in R^2, p^i \in R,\, s \in R^2. \end{aligned} \tag{20}$$

A3. Function $\sigma(\cdot)$ is positively homogeneous (satisfies condition (4)) and piecewise linear, that is, it is formed by gluing together a finite set of linear functions

$$\sigma^i(x) =< s^i, x >, \quad i \in \overline{1, n_\sigma}, \quad s^i \in R^2,\, x \in R^2.$$

Denote

$$Z = \{s^i | i \in \overline{1, n_\sigma}\}. \tag{21}$$

Moreover, by virtue of representations (17), (18), without loss of generality we can assume that the function $\sigma(\cdot)$ is nonnegative

$$\sigma(x) \geq 0, \quad x \in R^2, \tag{22}$$

and consider the algorithm for constructing the function $\varphi(\cdot)$, corresponding to the limit function

$$\sigma^\sharp(x,y) = \sigma(x) + y_+, \quad x \in R^2,\, y \in R. \tag{23}$$

4.3 Simple piecewise linear functions

In Subbotin and Shagalova (1992); Shagalova (1999) the concept of simple piecewise linear functions (SPLF) was used. Here this concept will also be useful. The main property of an SPLF is the following. If $\psi(\cdot)$ is an SPLF, then for an arbitrary point $x_* \in R^2$ in its domain, there is a neighbourhood $O_\varepsilon(x_*)$ where $\psi(\cdot)$ has one of three possible representations:

$$\psi(x) =< s_i, x > + h_i,$$
$$\psi(x) = \max\{< s_i, x > + h_i, < s_j, x > + h_j >\},$$
$$\psi(x) = \min\{< s_i, x > + h_i, < s_j, x > + h_j >\}.$$

Here s_i and s_j are vectors in R^2, and h_i and h_j are numbers. Thus, the domain of definitions of an SPLF contains no points in small neighbourhoods of which three or more linear functions are sewn together.

Structural matrices may be used for formal definition of SPLFs. The structural matrix (SM) contains an information about all linear functions, that are forming the corresponding SPLF. Given the SM, one can easy calculate the value of the corresponding SPLF in every point in its domain.

Remark. If condition **A3** is satisfied, the nonnegative function $\sigma : R^2 \longrightarrow R$ is SPLF in domain $R^2 \setminus 0$, where 0 is zero vector.

4.4 Elementary problems

The algorithm for constructing the function $\varphi(\cdot)$ consists, essentially, in the sequential solution of elementary problems that arise in a certain order. These problems can be formulated as follows.

Let

$$\varsigma^{\sharp+}(x,y) = \max\{< a, x > + y, < b, x > + y\},$$
$$\varsigma^{\sharp-}(x,y) = \min\{< a, x > + y, < b, x > + y\},$$

where a, b, x are vectors from R^2, $y \in R$.

P r o b l e m s 1 and 2. Let some linearly independent vectors $a \in R^2$ and $b \in R^2$ be given. In problem 1 [problem 2] it is required to construct the minimax solution of problem (13), (14), (20) with $\sigma^\sharp = \varsigma^{\sharp+}$ (with $\sigma^\sharp = \varsigma^{\sharp-}$).

Since the function $\varsigma^{\sharp+}$ is convex, and the function $\varsigma^{\sharp-}$ is concave, one can obtain explicit formulas for solutions of problems 1 and 2. It is not difficult to verify that the solutions of these problems are functions

$$\phi^+(x,y) = \max_{l \in [a,b]} \phi_l(x,y), \quad \phi^-(x,y) = \min_{l \in [a,b]} \phi_l(x,y),$$

where

$$[a,b] = \{\lambda a + (1-\lambda) b \,|\, \lambda \in [0,1]\},$$
$$\phi_l(x,y) =< l, x > + y + H(l).$$

The first stage of the algorithm for constructing the solution $\varphi(\cdot)$ of problem (13), (14), (23) consists in solving problems 1 and 2. Specific problems that need to be solved are determined by the function $\sigma(\cdot)$.

The further construction of the solution consists in solving elementary problems of a different type.

Let $\bar{s} = (s_1, s_2, s_3) \in R^3$. Denote

$$\varphi_{\bar{s}}(x,y) =< s, x > + s_3 \cdot y + H^*(\bar{s}), \quad x \in R^2,\, y \in R$$

where the vector $s \in R^2$ is formed by the first two components of the vector $\bar{s}$, $s = (s_1, s_2)$. Let us note that if the component $s_3 = 1$, then $H^*(\bar{s}) = H(s)$ and $\varphi_{\bar{s}}(x, y) = \phi_s(x, y)$.

For given set M we denote its closure by the symbol clM and its boundary by the symbol ∂M.

P r o b l e m s 3 and 4. Let some linearly independent vectors $\bar{a} = (a_1, a_2, a_3) \in R^3$ and $\bar{b} = (b_1, b_2, b_3) \in R^3$ and a number $r > 0$ be given. Let

$$\varphi^*(x, y) = \max\{\varphi_{\bar{a}}(x, y),\ \varphi_{\bar{b}}(x, y)\},$$
$$\varphi_*(x, y) = \min\{\varphi_{\bar{a}}(x, y),\ \varphi_{\bar{b}}(x, y)\},$$
$$G^* = \{(x, y) \in R^3 | \varphi^*(x, y) < r\},$$
$$G_* = \{(x, y) \in R^3 | \varphi_*(x, y) < r\}.$$

In problem 3 it is required to construct a continuous function $\varphi^0 : clG^* \to R$ which is a minimax solution of the PDE (13) in G^* and satisfies the relations

$$\varphi^0(x, y) < r,\ \forall (x, y) \in G^*; \varphi^0(x, y) = r,\ \forall (x, y) \in \partial G^*.$$

In problem 4 it is required to construct a continuous function $\varphi_0 : clG_* \to R$ which is a minimax solution of the PDE (13) in G_* and satisfies the relations

$$\varphi_0(x, y) < r,\ \forall (x, y) \in G_*; \varphi_0(x, y) = r,\ \forall (x, y) \in \partial G_*.$$

The solution of Problem 3 is the function

$$\varphi^0(x, y) = \max_{\bar{s}} \varphi_{\bar{s}}(x, y) \quad \text{for} \quad \bar{s} \in S_r(\bar{a}, \bar{b}),$$

where

$$S_r(\bar{a}, \bar{b}) = \{\bar{s} \in con(\bar{a}, \bar{b}) | < s, w_0 > +H^*(\bar{s}) = r\},$$
$$con(\bar{a}, \bar{b}) = \{\lambda\bar{a} + \mu\bar{b}) | \lambda \geq 0, \mu \geq 0\},$$

and point $w_0 \in R^2$ is the solution of the system of two linear equations

$$< a, w_0 > +H^*(\bar{a}) = r, \quad < b, w_0 > +H^*(\bar{b}) = r.$$

Components of vectors $a \in R^2$ and $b \in R^2$ coincide with the first two components of $\bar{a}$ and $\bar{b}$, respectively.

The solution of Problem 4 in the cases which arise in the construction of the solution $\varphi(\cdot)$ of problem (13), (14), (23) is the function

$$\varphi_0(x, y) = \min_{\bar{s}} \varphi_{\bar{s}}(x, y) \quad \text{for} \quad \bar{s} \in S_r(\bar{a}, \bar{b}).$$

5. THE MAIN RESULT

Let us denote by Ω the set of points in the space R^3 where the Hamiltonian $H^* : R^3 \to R$ is nondifferentiable. Let 0 be the zero vector in R^3. By the set $Z \subset R^2$ (21) of vectors forming function σ, we define the set $Z^\natural \subset R^3$

$$Z^\natural = \{\bar{s} = (s_1, s_2, s_3) \in R^3 | s = (s_1, s_2) \in Z, s_3 = 1\}.$$

Now we can formulate an assertion containing the main result of the paper.

Theorem 2. Suppose that conditions **A2-A3** are satisfied. Then

A) The solution $\varphi(\cdot)$ of problem (13), (14), (23) is nonnegative function formed by sewing together linear functions

$$\varphi_{\bar{s}}(x, y) =< s, x > +s_3 \cdot y + H^*(\bar{s}), \quad \bar{s} \in L,$$

where the set L consists of a finite number of elements, and

$$Z^\natural \subset L, \quad (L \setminus Z^\natural) \subset (\Omega \cup 0).$$

B) For every $y_* \in R$ function $\varphi(x, y_*)$ in the region $\{x \in R^2 | \varphi(x, y_*) > 0\}$ is formed by sewing together a finite collection of simple piecewise linear functions.

Proof. The assertion of the theorem follows from the proposed algorithm for constructing the solution. The function $\varphi(\cdot)$ is glued together from solutions of elementary problems, so inequalities (15), (16) are satisfied. Omitting the detailed description, we show here only that the function $\varphi(\cdot)$ satisfies the limit relation (14).

Consider an arbitrary point $(x_*, y_*), x_* \in R^2$, $y_* \in R$, $y_* \geq 0$.

If x_* is the zero vector, then for any $\alpha > 0$ the vector $\frac{x_*}{\alpha}$ is also zero. In the case $y_* = 0$ we have

$$\lim_{\alpha \downarrow 0} \alpha\varphi\left(\frac{0}{\alpha}, \frac{0}{\alpha}\right) = \lim_{\alpha \downarrow 0} \alpha\varphi(0, 0) = 0 = \sigma^\sharp(0, 0).$$

In the case $y_* \neq 0$ it follows from the algorithm that there exist a number $\alpha_* > 0$ and a vector $s \in R^2$ such that for all $0 < \alpha < \alpha_*$ the following relation holds.

$$\varphi\left(0, \frac{y_*}{\alpha}\right) = \frac{y_*}{\alpha} + H(s).$$

From this relation we obtain

$$\lim_{\alpha \downarrow 0} \alpha\varphi\left(\frac{0}{\alpha}, \frac{y_*}{\alpha}\right) = y_* = \sigma^\sharp(0, y_*).$$

Now let x_* be a nonzero vector. If there exists a neighborhood $O(x_*)$ of the point x_* on the plane R^2 in which the function $\sigma(\cdot)$ is linear

$$\sigma(x) = \langle a, x \rangle, \quad x \in O(x_*),$$

then it follows from the algorithm that there exists a number $\alpha_* > 0$ such that for all $0 < \alpha < \alpha_*$

$$\varphi\left(x, \frac{y_*}{\alpha}\right) = \left\langle a, \frac{x}{\alpha} \right\rangle + \frac{y_*}{\alpha} + H(a).$$

Thus,

$$\lim_{\alpha \downarrow 0} \alpha\varphi\left(\frac{x_*}{\alpha}, \frac{y_*}{\alpha}\right) = \langle a, x_* \rangle + y_* = \sigma^\sharp(x_*, y_*).$$

It remains to consider the case when the function $\sigma(\cdot)$ is glued together from two linear functions in the neighborhood $O(x_*)$. Suppose, for definiteness, the linear functions are glued together by the operation of the maximum

$$\sigma(x) = \max\{\langle a, x \rangle, \langle b, x \rangle\} \quad x \in O(x_*).$$

Then there exists a number $\alpha_* > 0$ such that for all $0 < \alpha < \alpha_*$ the point $\left(\frac{x_*}{\alpha}, \frac{y_*}{\alpha}\right)$ is in the region in which the function φ coincides with the solution of the first elementary problem defined by the vectors a and b, and

$$\varphi\left(\frac{x_*}{\alpha}, \frac{y_*}{\alpha}\right) =$$
$$= \max_{\lambda \in [0,1]} \left\{\left\langle \lambda a + (1 - \lambda)b, \frac{x_*}{\alpha} \right\rangle + \frac{y_*}{\alpha} + H(\lambda a + (1 - \lambda)b)\right\}.$$

We get

$$\lim_{\alpha \downarrow 0} \alpha\varphi\left(\frac{x_*}{\alpha}, \frac{y_*}{\alpha}\right) = \max\{\langle a, x \rangle, \langle b, x \rangle\} + y = \sigma^\sharp(x_*, y_*),$$

which completes the verification of the fulfillment of the limit relation (14).

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